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Scattering by the Singular Potential $-gr^{-4}\ln r$

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In connection with higher order leptonic weak interactions, divergent series containing an infinite number of logarithmic factors are encountered. In order to have some understanding of series of this type, the problem of scattering by the singular potential $-gr^{-4}\ln r$ is examined for the special case of zero energy. This problem has the advantage that the correct answer can be obtained by other means. Generalization to the potential $-gr^{-\lambda}\ln r$ with $\lambda>3$ is also considered.

1. INTRODUCTION

RECENTLY, attempts have been made to obtain some understanding of the corrections to processes involving weak interactions. Lee and Yang[1] have developed for this purpose the ξ-limiting formalism for the intermediate boson, and Feinberg and Pais[2] have studied higher order weak interactions restricted to the uncrossed ladder diagrams. In both cases, summation of divergent power series is a necessary step. As a possible check on this procedure of summing divergent power series, Khuri and Pais[3] and Tiktopoulos and Treiman[4] considered the exactly solvable problem of scattering at zero energy by the singular power potential $gr^{-\lambda}$, where $\lambda>3$. In this case, summing the Born series after first introducing a cutoff does indeed give the correct answer. However, the relevance of this model to field theory is by no means clear.

In connection with higher order leptonic weak interactions, it has been proposed[5] to sum a divergent series involving logarithmic factors. Although this procedure does yield a definite answer taking into account all possible Feynman-Dyson diagrams, summing such a series must be considered to be even more dubious than summing a divergent power series of the kind mentioned above. Indeed, while divergent power series have been encountered in various branches of physics, divergent series with an infinite number of logarithmic factors seem to be something novel. It is therefore the purpose of this paper to study a potential problem where a series of this variety appears. For the sake of definiteness, we consider first the scattering by the potential $-gr^{-4}\ln r$, where $g>0$ and hence the potential is repulsive near the origin. Some generalization to other powers is discussed in Appendix E. As in the earlier works,[3,4] we restrict ourselves to the case of zero energy.

The procedure is as follows. After introducing a cutoff Λ^{-1} in the radial coordinate, we expand the scattering length in powers of the coupling constant g, with coefficients that depend on various powers of Λ and $\ln\Lambda$. In Sec. 2, we rearrange the various terms and define the series $B_n(\Lambda)$ to be summed. In Sec. 3, we study the two simplest series by a method that can partially be generalized to all n, as shown in Sec. 4. The result is that none of the $B_n(\Lambda)$ approach any limit for the potential without a cutoff. Some properties of these functions $B_n(\Lambda)$ are given in Sec. 5 together with a comparison with the case of higher order leptonic weak interactions mentioned above.

2. FORMULATION OF THE PROBLEM

We consider the radial differential equation

$$d^2\psi/dr^2+gr^{-4}(\ln r)\psi=0\,, \tag{2.1}$$

with the boundary conditions $\psi(0)=0$ and $\psi(r)\sim r$ as $r\to\infty$. The scattering length A is defined by

$$A=\lim_{r\to\infty}[\psi(r)-r]. \tag{2.2}$$

In order to use Born series, we introduce a cutoff Λ^{-1}. Thus, (2.1) is replaced by

$$d^2\psi(r,\Lambda)/dr^2=\begin{cases}-gr^{-4}\ln r\psi(r,\Lambda)\,, & \text{for } r>\Lambda^{-1}\,,\\ 0\,, & \text{for } r<\Lambda^{-1}\,,\end{cases} \tag{2.3}$$

with the boundary conditions $\psi(0,\Lambda)=0$ and $\psi(r,\Lambda)\sim r$ as $r\to\infty$. Similarly define

$$A(\Lambda)=\lim_{r\to\infty}[\psi(r,\Lambda)-r]. \tag{2.4}$$

Equation (2.3) together with the boundary conditions is equivalent to the integral equation

$$\psi(r,\Lambda)=r+g\int_{\Lambda^{-1}}^{\infty}dr'\,r'^{-4}\ln r'\min(r,r')\psi(r',\Lambda)\,, \tag{2.5}$$

which may be solved formally by iteration. Let

$$\psi_0(r,\Lambda)=r\,, \tag{2.6}$$

and

$$\psi_{n+1}(r,\Lambda)=g\int_{\Lambda^{-1}}^{\infty}dr'r'^{-4}\ln r'\min(r,r')\psi_n(r',\Lambda) \tag{2.7}$$

for $n\geqslant 0$. According to (2.4), (2.6), and (2.7), define $A_0(\Lambda)=0$ and, for $n>1$,

$$A_n(\Lambda)=\psi_n(\infty,\Lambda)=g\int_{\Lambda^{-1}}^{\infty}dr'\,r'^{-3}\ln r'\psi_{n-1}(r',\Lambda)\,. \tag{2.8}$$

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In view of (2.6–2.8), $A_n(\Lambda)$ must be of the following form:

$$A_n(\Lambda)=g^n\Lambda^{2n-1}\sum_{m=0}^{n} a_{nm}(\ln\Lambda)^{n-m}. \tag{2.9}$$

Let

$$B_m(\Lambda)=\sum_{n=m}^{\infty} a_{nm}g^n\Lambda^{2n-1}(\ln\Lambda)^{n-m}, \tag{2.10}$$

then formally

$$A(\Lambda)=\sum_{n=1}^{\infty} A_n(\Lambda)=\sum_{m=0}^{\infty} B_m(\Lambda). \tag{2.11}$$

It is the purpose of this paper to study in detail the behavior of $B_m(\Lambda)$ for large Λ. For the sake of orientation and later comparison, in Appendix A we compute approximately the scattering length A for small g.

3. PROPERTIES OF THE FIRST TWO TERMS

In this section, we compute explicitly $B_1(\Lambda)$ and $B_2(\Lambda)$. In order that at least a portion of the consideration can be generalized to all $B_n(\Lambda)$, we follow a somewhat devious sequence of steps. The more straightforward calculation, which involves no less algebraic manipulation, is relegated to Appendix B.

We begin with a relation between the scattering length A and the Jost function,[6] as given by (C5) of Appendix C:

$$A(\Lambda)=\int_0^\infty dr\{[f(r,\Lambda)]^{-2}-1\}, \tag{3.1}$$

where $f(r,\Lambda)$ is the Jost function for the potential cutoff at Λ^{-1}. Let $f(r)=f(r,\infty)$, then $f(r,\Lambda)$ has the important property

$$f(r,\Lambda)=f(r) \tag{3.2}$$

for $r>\Lambda^{-1}$.

It is convenient to use the variable

$$x=r^{-1}, \tag{3.3}$$

and define

$$F(x,\Lambda)=r^{-1}f(r,\Lambda), \tag{3.4}$$

$$F(x)=r^{-1}f(r). \tag{3.5}$$

It follows from (3.1) and the differential equation for $f(r,\Lambda)$ that

$$A(\Lambda)=-g\int_0^\Lambda dx\,\ln x\left[\frac{d}{dx}F(x,\Lambda)\right]^{-2},$$

which, by (3.2) and (3.5), can be simplified to

$$A(\Lambda)=-g\int_0^\Lambda dx\,\ln x[F'(x)]^{-2}. \tag{3.6}$$

A comparison with the calculation of Appendix B shows that we have avoided the simultaneous treatment of two variables by using the Jost function.

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We proceed to calculate $F(x)$ and $F'(x)$ approximately. The function $F(x)$ satisfies the integral equation

$$F(x)-g\int_0^x dx'(x-x')\ln x'F(x')=x. \tag{3.7}$$

We formally iterate this equation by defining

$$F_0(x)=x, \tag{3.8}$$

and

$$F_{n+1}(x)=g\int_0^x dx'(x-x')\ln x'F_n(x') \tag{3.9}$$

for $n\geq 0$. It is then seen that $F_n(x)$ is the product of x^{2n+1} with a polynomial of order n in $\ln x$. We shall keep only the two leading terms:

$$F_n(x)\sim g^n x^{2n+1}[M_n(\ln x)^n+N_n(\ln x)^{n-1}]. \tag{3.10}$$

With the formula

$$\int dx\,x^{p-1}(\ln x)^q =p^{-1}x^p\sum_{k=0}^{q}(-1)^{q-k}p^{-q+k}(\ln x)^k q!/k! \tag{3.11}$$

for any non-negative integer q, the substitution of (3.10) into (3.9) gives the recurrence relations

$$M_{n+1}=[(2n+2)^{-1}-(2n+3)^{-1}]M_n, \tag{3.12}$$

and

$$N_{n+1}=[(2n+2)^{-1}-(2n+3)^{-1}]N_n -(n+1)[(2n+2)^{-2}-(2n+3)^{-2}]M_n. \tag{3.13}$$

The solutions of (3.12) and (3.13) with the boundary conditions $M_0=1$ and $N_0=0$ are

$$M_n=[(2n+1)!]^{-1}, \tag{3.14}$$

and

$$N_n=-[(2n+1)!]^{-1}[n-\tfrac{1}{2}\sum_{k=0}^{n}(2k+1)^{-1}]. \tag{3.15}$$

Summation over n gives, with the help of (B20),

$$F(x)\sim x\left\{z^{-1}\sinh z-(2\ln x)^{-1}\left[\cosh z -z^{-1}\int_0^z dz'z'^{-1}\sinh z'\cosh(z-z')\right]\right\}, \tag{3.16}$$

where

$$z=(gx^2\ln x)^{1/2}. \tag{3.17}$$

The derivative of $F(x)$ is obtained from

$$F'(x)=dF(x)/dx=\partial F(x)/\partial x +z[1+(2\ln x)^{-1}]x^{-1}\partial F(x)/\partial z; \tag{3.18}$$

and the result is

$$F'(x)\sim\cosh z-(2\ln x)^{-1}\times\left[z\sinh z-\int_0^z dz'z'^{-1}\sinh z'\sinh(z-z')\right]. \quad (3.19)$$

It remains to substitute (3.19) into (3.6). By (3.11), we can obtain $B_0(\Lambda)$ by using the first term of (3.19) and considering $\ln x$ to be a constant; more precisely:

$$B_0(\Lambda)=-g\int_0^\Lambda dx\,\omega[\cosh(g^{1/2}\omega^{1/2}x)]^{-2}|_{\omega=\ln\Lambda}, \quad (3.20)$$

where ω is taken to be a constant in performing the integration over x. Evaluation of the right-hand side of (3.20) leads to

$$B_0(\Lambda)=-g^{1/2}(\ln\Lambda)^{1/2}\tanh(g\Lambda^2\ln\Lambda)^{1/2}. \quad (3.21)$$

The computation of $B_1(\Lambda)$ is far more complicated. Equation (3.11) may be written in the following form for $p>0$:

$$\int_0^x dx'x'^{p-1}(\ln x')^q=\sum_{k=0}^{\infty}\left[(-1)^k\left(\frac{\partial}{\partial\omega}\right)^k S^k\int_0^x dx'x'^{p-1}\omega^q\right]\bigg|_{\omega=\ln x}, \quad (3.22)$$

where S is the integration operation

$$Sh(x)=\int_0^x dx'h(x')/x'. \quad (3.23)$$

In the form (3.22), on the right-hand side, p and q appear only in the integrand. Therefore, with suitable conditions on $\mathfrak{F}$,

$$\int_0^x dx'\mathfrak{F}(x',\ln x')=\sum_{k=0}^{\infty}\left[(-1)^k\left(\frac{\partial}{\partial\omega}\right)^k S^k\int_0^x dx'\mathfrak{F}(x',\omega)\right]\bigg|_{\omega=\ln x}. \quad (3.24)$$

Equation (3.24) may be used to find $B_1(\Lambda)$. By (3.19), (3.6), (2.11), and (3.21), $B_1(\Lambda)$ consists of two parts:

$$B_1(\Lambda)=B_1^{(1)}(\Lambda)+B_1^{(2)}(\Lambda), \quad (3.25)$$

where

$$B_1^{(1)}(\Lambda)=\left[(\partial/\partial\omega)\int_0^\Lambda dx\,x^{-1}(g\omega)^{1/2}\times\tanh(g\omega x^2)^{1/2}\right]\bigg|_{\omega=\ln\Lambda}; \quad (3.26)$$

and

$$B_1^{(2)}(\Lambda)=-g\left(\int_0^\Lambda dx\,\operatorname{sech}^3(g\omega x^2)^{1/2}\times\left\{(g\omega x^2)^{1/2}\sinh(g\omega x^2)^{1/2}-\int_0^{(g\omega)^{1/2}x}dz'z'^{-1}\times\sinh z'\sinh[(g\omega)^{1/2}x-z']\right\}\right)\bigg|_{\omega=\ln\Lambda}. \quad (3.27)$$

It follows from (3.26) and (3.27) that

$$B_1^{(1)}(\Lambda)=\tfrac{1}{2}g^{1/2}(\ln\Lambda)^{-1/2}\left[\tanh(g\Lambda^2\ln\Lambda)^{1/2}+\int_0^\Lambda dx\,x^{-1}\tanh(gx^2\ln\Lambda)^{1/2}\right], \quad (3.28)$$

and

$$B_1^{(2)}(\Lambda)=-g^{1/2}(\ln\Lambda)^{-1/2}\int_0^{(g\Lambda^2\ln\Lambda)^{1/2}}dz\,\operatorname{sech}^3 z\times\left[z\sinh z-\int_0^z dz'z'^{-1}\sinh z'\sinh(z-z')\right], \quad (3.29)$$

or

$$B_1(\Lambda)=\tfrac{1}{2}g^{1/2}(\ln\Lambda)^{-1/2}\operatorname{sech}\Omega\times\left[\Omega\operatorname{sech}\Omega+\int_0^\Omega dz z^{-1}\sinh z\cosh(\Omega-z)+\tanh\Omega\int_0^\Omega dz z^{-1}\sinh z\sinh(\Omega-z)\right], \quad (3.30)$$

where

$$\Omega=(g\Lambda^2\ln\Lambda)^{1/2}. \quad (3.31)$$

This is the desired answer.

Asymptotically, as $\Lambda\to\infty$,

$$B_0(\Lambda)=-g^{1/2}(\ln\Lambda)^{1/2}+o(1), \quad (3.32)$$

and

$$B_1(\Lambda)=\tfrac{1}{2}g^{1/2}(\ln\Lambda)^{1/2}+o(1). \quad (3.33)$$

4. ASYMPTOTIC BEHAVIORS OF $B_n(\Lambda)$

It is seen from (3.30) that $B_1(\Lambda)$ is a rather complicated function. Indeed, it is so much more complicated than $B_1(\Lambda)$ that one shudders at the task of finding $B_2(\Lambda)$. Yet the asymptotic behavior of $B_1(\Lambda)$, as given by (3.33), is extremely simple. In this section, we generalize (3.32)–(3.33) to all $B_n(\Lambda)$.

For this purpose, we study the asymptotic behavior of $B_1(\Lambda)$ in some detail. Since the integral on the right-hand side of (3.29) approaches a finite limit as $\Lambda\to\infty$ because of the factor $\operatorname{sech}^3 z$ in the integrand, we have

$$B_1^{(2)}(\Lambda)=O((\ln\Lambda)^{-1/2}) \quad (4.1)$$

as $\Lambda\to\infty$. This is a direct consequence of the fact that, in (3.19), the first term on the right-hand side is of the order of e^z, while the second term is of the order of ze^z, as $z\to\infty$. More generally, the nth term of $F'(x)$, that is, the coefficient of $(\ln x)^{-n}$, is of the order of $z^n e^z$. Therefore,

$$B_n(\Lambda)=B_n^{(1)}(\Lambda)+O((\ln\Lambda)^{-1/2}), \tag{4.2}$$

where $B_n^{(1)}(\Lambda)$ is given by a generalization of (3.26) by (3.24):

$$B_n^{(1)}(x)=(-1)^{n+1}\times[(\partial/\partial\omega)^n S^n(g\omega)^{1/2}\tanh(g\omega x^2)^{1/2}]|_{\omega=\ln x}. \tag{4.3}$$

As $x\to\infty$, (4.3) gives, by dropping the hyperbolic tangent factor, as $\Lambda\to\infty$

$$B_n^{(1)}(\Lambda)=(-1)^{n+1}C_n^{1/2}g^{1/2}(\ln\Lambda)^{1/2}+O((\ln\Lambda)^{-1/2}), \tag{4.4}$$

where $C_n^{1/2}$ is the binomial coefficient. Consequently,

$$B_n(\Lambda)=(-1)^{n+1}C_n^{1/2}g^{1/2}(\ln\Lambda)^{1/2}+O((\ln\Lambda)^{-1/2}) \tag{4.5}$$

as $\Lambda\to\infty$.

5. DISCUSSIONS

The present situation differs greatly from that of a power potential, where no approximation is made in the sense that all terms are kept throughout the manipulation.[3,4] Indeed, since all terms are kept, it is difficult to imagine any failure to obtain the correct answer. This is not at all the case here, and it may be of some interest to note that none of the $B_n(\Lambda)$ approach a finite limit as $\Lambda\to\infty$. However, observe that essentially the correct answer is obtained if only B_0 is retained with Λ reinterpreted as $g^{-1/2}$. Secondly, we may try to sum the right-hand side of (4.5) over all n. Since by the binomial theorem

$$\sum_{n=0}^{N}(-x)^n C_n^{1/2}-(1-x)^{1/2}=2^{-2N-1}(N!)^{-2}(2N)!\times\int_0^x dx'(1-x')^{-1/2-n}(x-x')^N, \tag{5.1}$$

we have, by setting $x=1$,

$$\sum_{n=0}^{N}(-1)^n C_n^{1/2}=2^{-2N}(N!)^{-2}(2N)!. \tag{5.2}$$

Hence, by Sterling's formula

$$\sum_{n=0}^{\infty}(-1)^n C_n^{1/2}=0. \tag{5.3}$$

It is perhaps gratifying to find that a zero appears in the sum of the asymptotic formulas of $B_n(\Lambda)$ to avoid a result that is unambiguously infinite.

Thirdly, let us define the sum

$$\hat{A}=\lim_{\Lambda\to\infty}\sum_{n=0}^{\infty}B_n^{(1)}(\Lambda). \tag{5.4}$$

It follows from (3.24), (3.21), and (3.20) that

$$\hat{A}=-g\int_0^\infty dx(\ln x)[\cosh(gx^2\ln x)^{1/2}]^{-2}, \tag{5.5}$$

which is evaluated approximately for small g in Appendix D. The result is

$$\hat{A}=-g^{1/2}\epsilon^{-1/2}[1-\tfrac{1}{2}\epsilon(2\ln2+\gamma+1-\ln\pi)+O(\epsilon^2)]. \tag{5.6}$$

where ϵ is defined by (A3)–(A4). Numerically,

$$\hat{A}=-g^{1/2}\epsilon^{-1/2}[1-0.9094\epsilon+O(\epsilon^2)]. \tag{5.7}$$

while (A9) gives

$$A=-g^{1/2}\epsilon^{-1/2}[1-0.6352\epsilon+O(\epsilon^2)]. \tag{5.8}$$

Thus for small g, the right-hand side of (5.5) gives the first term correctly, while the second term is off by nearly 50%.

Fourthly, the question may be raised whether the results depend on the rather arbitrary definition of the cutoff Λ. For example, we may choose to use instead

$$\bar{\Lambda}=\Lambda/\sigma, \tag{5.9}$$

where σ is a positive constant. Analogous to (2.9) and (2.10), we may expand the scattering length in terms of $\bar{\Lambda}$ and $\ln\bar{\Lambda}$:

$$A_n(\Lambda)=g^n\bar{\Lambda}^{2n-1}\sum_{m=0}^{n}\bar{a}_{nm}(\ln\bar{\Lambda})^{n-m}, \tag{5.10}$$

and

$$\bar{B}_m(\Lambda)=\sum_{n=m}^{\infty}\bar{a}_{nm}g^n\bar{\Lambda}^{2n-1}(\ln\bar{\Lambda})^{n-m}. \tag{5.11}$$

Since $\bar{B}_n(\Lambda)$ may be obtained from $B_n(\Lambda)$ by Taylor's expansion, and

$$d(\ln\Lambda)^{1/2}/d(\ln\Lambda)\to0$$

as $\Lambda\to\infty$, $\bar{B}_n(\Lambda)$ also satisfies (4.5); i.e.,

$$\bar{B}_n(\Lambda)=(-1)^{n+1}C_n^{1/2}g^{1/2}(\ln\Lambda)^{1/2}+O((\ln\Lambda)^{-1/2})$$

as $\Lambda\to\infty$.

Finally, since the motivation for the present investigation comes originally from field theory, it remains to discuss the relevance of this model, or the lack thereof, to unrenormalizable field theory. We shall not enter into the general, and presumably unanswerable, question of the relation between problems of potential scattering and field theories; instead, we shall restrict ourselves to a comparison of the present calculation with that of the previous consideration on higher order leptonic weak interactions.[5] Even if we make the questionable identification of Λ with $\xi^{-1/2}m$,[7] the following differences still come to mind immediately:

[7] The ξ used in Ref. 5 is the renormalized ξ. Since the scalar particle with mass $\xi^{-1/2}m$ is unstable against decay into vector mesons, the value of ξ cannot be real. The fact that ξ is not real seems to lead to numerous complications that remain to be studied.

(1) Each of the infinite series considered is of the form

$$(\Lambda^h)^\mu \times [\text{function of } (\Lambda^h \ln\Lambda)].$$

For $B_0(\Lambda)$, $\mu=-\frac{1}{2}$; for the previous case, $\mu=1$. In particular, (a) μ is negative here but is positive in the previous case, and (b) μ is an integer in the previous case but is not so here.

(2) In the limit $\Lambda\to\infty$, B_0 does not approach a finite limit, while G does.[5]

It seems very difficult to assess the importance of the difference (1). If we restrict ourselves to one power of r in the potential, then it is not hard to construct somewhat more complicated examples of scattering by a potential such that the difference (1a) is removed. Unfortunately, the removal of the difference (1b) seems to necessitate the introduction of $(\ln r)^2$ in the potential. Whether this is worth doing is very questionable, since the set of Feynman-Dyson diagrams taken into account in the previous work[5] is a rather complicated one and bears no conceivable relation to ladder diagrams or Born series.

On the other hand, the difference (2) is almost certainly of paramount importance. It should perhaps be emphasized that, in the previous consideration,[5] it is *not* a separate assumption that G approaches a finite limit as $\xi\to 0$, i.e., G has to approach the limit given there provided that F approaches a finite limit. And in the series for F, there is altogether only one logarithmic factor, which appears in the first term. Because of these differences, this model of scattering by a potential fails to throw any light on the problem of field theory. It is tempting to speculate that perhaps logarithmic factors may arise for very different reasons and accordingly have different effects on infinite series.

Generalization of the present consideration to the potential $-gr^{-\lambda}\ln r$ is considered in Appendix E.

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APPENDIX A

In this Appendix we solve (2.1) approximately for small g. With (3.3) and $\varphi(x)=r^{-1}\psi(r)$, (2.1) is equivalent to

$$d^2\varphi/dx^2-g(\ln x)\varphi=0\,, \tag{A1}$$

where the boundary conditions are $\varphi(0)=1$ and $\varphi(\infty)=0$. The scattering length is $A=d\varphi(x)/dx|_{x=0}$.

Suppose a scale transformation is carried out on the independent variable: $x=\tau y$. Then

$$d^2\varphi/dy^2-g\tau^2(\ln\tau+\ln y)\varphi=0\,. \tag{A2}$$

We choose τ such that

$$g\tau^2\ln\tau=1\,; \tag{A3}$$

then

$$d^2\varphi/dy^2-(1+\epsilon\ln y)\varphi=0\,,$$

where

$$\epsilon=(\ln\tau)^{-1}. \tag{A4}$$

Define

$$\bar\varphi(y)=\varphi(y)+\tfrac{1}{2}\epsilon\int_0^\infty dy'\varphi(y')e^{-|y-y'|}\ln y'\,; \tag{A5}$$

then $\bar\varphi(y)$ satisfies $d^2\bar\varphi/dy^2-\bar\varphi=0$ together with the boundary condition $\bar\varphi(\infty)=0$. Thus, $\bar\varphi(y)=Ce^{-y}$ and

$$\varphi(y)=Ce^{-y}-\tfrac{1}{2}\epsilon\int_0^\infty dy'\varphi(y')e^{-|y-y'|}\ln y'. \tag{A6}$$

The constant C is given by

$$C=1+\tfrac{1}{2}\epsilon\int_0^\infty dy\varphi(y)e^{-y}\ln y\,, \tag{A7}$$

and A by

$$A=-\tau^{-1}(2C-1)\,. \tag{A8}$$

Finally, iteration of (A6) gives that

$$A=-g^{1/2}\epsilon^{-1/2}\{1-\tfrac{1}{2}\epsilon(\ln 2+\gamma) -\tfrac{1}{8}\epsilon^2[\tfrac{1}{3}\pi^2+(\ln 2+\gamma-1)^2-3]+O(\epsilon^3)\}\,, \tag{A9}$$

where γ is Euler's constant.

The physical content of this approximate procedure is as follows. Since $\ln r$ is a slowly varying function, we approximate our potential $-gr^{-4}\ln r$ by $g'r^{-4}$, whose scattering length is $-(g')^{1/2}$. We determine g' by requiring the two potentials to be equal at $r=(g')^{1/2}$. Thus

$$-g\ln(g')^{1/2}=g'\,, \tag{A10}$$

or

$$g'=\tau^{-2}=g\ln\tau\,. \tag{A11}$$

Similar considerations may be applied to other problems of scattering by a potential which differs from a strongly singular repulsive potential by a slowly varying factor.

APPENDIX B

In this Appendix, we derive (3.21) and (3.30) directly from the ψ_n of (2.6) and (2.7). Similarly to (3.4), let

$$\varphi_n(x,\Lambda)=r^{-1}\psi_n(x,\Lambda)\,, \tag{B1}$$

then

$$\varphi_0(x,\Lambda)=1\,, \tag{B2}$$

and

$$\varphi_{n+1}(x,\Lambda)=-g\int_0^\Lambda dx'\ln x'\min(x,x')\varphi_n(x',\Lambda) \tag{B3}$$

for $n\geqslant 0$. It also follows from (2.8) that

$$A_n(\Lambda)=\lim_{x\to 0}x^{-1}\varphi_n(x,\Lambda)=(d/dx)\varphi_n(x,\Lambda)|_{x=0}. \tag{B4}$$

To get $B_0(\Lambda)$ and $B_1(\Lambda)$, it is sufficient to approximate (2.9) by

$$A_n(\Lambda)\sim g^n\Lambda^{2n-1}(\ln\Lambda)^n[a_{n0}+a_{n1}(\ln\Lambda)^{-1}]. \tag{B5}$$

Accordingly, we approximate $\varphi_n(x,\Lambda)$ by

$$\varphi_n(x,\Lambda)\sim g^n\{\alpha_n x^{2n}(\ln x)^n[1+\xi_n(\ln x)^{-1}] + \sum_{k=0}^{n-1}\Lambda^{2k+1}x^{2n-2k-1}(\ln\Lambda)^{k+1}(\ln x)^{n-k-1} \times[\beta(k,n)+\beta(k,n)\rho(k,n)(\ln x)^{-1} +\eta(k,n)(\ln\Lambda)^{-1}]\}. \quad (B6)$$

In (B5) and (B6), $a_{01}=\zeta(n-1,n)=0$. By (B4),

$$\beta(n-1,n)=a_{n0} \quad\text{and}\quad \eta(n-1,n)=a_{n1}. \quad (B7)$$

The substitution of (B6) into (B3) gives, by (3.11),

$$\alpha_n=(2n!)^{-1}, \quad (B8)$$

$$\xi_{n+1}=\xi_n-(4n+3)/(4n+2), \quad (B9)$$

$$\beta(k,n)=[(2n-2k-1)!]^{-1}a_{k+1,0}, \quad (B10)$$

$$\rho(k,n+1)=\rho(k,n)-\tfrac{1}{2}(2n-2k+1)^{-1} \times(4n-4k+1), \quad (B11)$$

$$\eta(k,n)=a_{k+1,1}[(2n-2k-1)!]^{-1}, \quad (B12)$$

$$\sum_{k=0}^{n}(2n-2k)!a_{k+1,0}=-[(2n+1)!]^{-1}, \quad (B13)$$

and

$$a_{n+1,1}=-[(2n+1)!]^{-1}[\xi_n-(n+1)/(2n+1)] -\sum_{k=0}^{n-1}(2n-2k)^{-1}\{\beta(k,n)[\rho(k,n)-\tfrac{1}{2}]+\eta(k,n)\}, \quad (B14)$$

where (B10)–(B12) hold for $k<n$. Equation (3.21) follows immediately from (B13).

Let

$$\sigma_n=\sum_{j=1}^{n}(2j-1)^{-1}, \quad (B15)$$

then it follows from (B9) and (B11) that

$$\xi_n=-n-\tfrac{1}{2}\sigma_n \quad (B16)$$

and

$$\rho(k,n)=-(n-k-\tfrac{1}{2})+\tfrac{1}{2}\sigma_{n-k}. \quad (B17)$$

The substitution of (B10) and (B12) into (B14) gives

$$\sum_{k=0}^{n}[(2n-2k)!]^{-1}[a_{k+1,1}-a_{k+1,0}(n-k-\tfrac{1}{2}\sigma_{n-k})] =[(2n+1)!]^{-1}[n+\tfrac{1}{2}\sigma_n+(n+1)/(2n+1)]. \quad (B18)$$

Since

$$\sum_{n=0}^{\infty}\Omega^{2n}\sigma_n/(2n)!=\int_0^{\Omega}dzz^{-1}\sinh z\sinh(\Omega-z), \quad (B19)$$

and

$$\sum_{n=0}^{\infty}\Omega^{2n+1}[\sigma_n+(2n+1)^{-1}]/(2n+1)! =\int_0^{\Omega}dzz^{-1}\sinh z\cosh(\Omega-z), \quad (B20)$$

it follows from (B18) that

$$B_1(\Lambda)\cosh\Omega-\tfrac{1}{2}B_0(\Lambda)\left[\Omega\sinh\Omega -\int_0^{\Omega}dzz^{-1}\sinh z\sinh(\Omega-z)\right]\Big/\ln\Lambda =\tfrac{1}{2}g\Lambda\left[\cosh\Omega+\Omega^{-1}\int_0^{\Omega}dzz^{-1}\sinh z\cosh(\Omega-z)\right], \quad (B21)$$

where Ω is defined by (3.31). The substitution of (3.21) into (B21) gives (3.30).

APPENDIX C

We develop here a relation between the scattering length and the Jost function[6] at zero energy. At zero energy, the Jost function $f(r)$ and the associated $g(r)$ for a potential $V(r)$ may be defined by the integral equations[8]

$$f(r)-\int_r^{\infty}dr'(r'-r)V(r')f(r')=1, \quad (C1)$$

and

$$g(r)-\int_r^{\infty}dr'(r'-r)V(r')g(r')=r. \quad (C2)$$

Since $f(r)$ and $g(r)$ satisfy the same second-order linear ordinary differential equation, they can be expressed in terms of each other. For example, if $f(r)\neq 0$ for all r,

$$g(r)=f(r)\left(r-\int_r^{\infty}dr'\{[f(r')]^{-2}-1\}\right). \quad (C3)$$

Since the scattering length A is given by[8]

$$A=-\lim_{r\to 0}g(r)/f(r), \quad (C4)$$

we get from (C3)

$$A=\int_0^{\infty}dr\left\{[f(r)]^{-2}-1\right\}. \quad (C5)$$

We remark parenthetically that (C.5) is a special case of

$$A(k)=\int_0^{\infty}dr\left\{[f(-k,r)]^{-2}-e^{-2ikr}\right\}, \quad (C6)$$

where k^2 is the energy.

[8] A. Pais and T. T. Wu, Phys. Rev. **134**, B1303 (1964).

APPENDIX D

Here we evaluate (5.5) approximately. By (A3) and (A4), we have

$$\hat{A}=-g^{1/2}\epsilon^{-1/2}\int_0^\infty dy[1+\epsilon\ln y]\times[\cosh(1+\epsilon\ln y)^{1/2}y]^{-2}. \quad \text{(D1)}$$

Expansion in ϵ gives

$$\hat{A}=-g^{1/2}\epsilon^{-1/2}\left[1+\epsilon\int_0^\infty dy\ln y\,\mathrm{sech}^3 y\times(\cosh y-y\sinh y)+O(\epsilon^2)\right]$$
$$=-g^{1/2}\epsilon^{-1/2}\left[1-\frac{\epsilon}{2}\left(1-\int_0^\infty dy\ln y\,\mathrm{sech}^2 y\right)+O(\epsilon^2)\right]. \quad \text{(D2)}$$

This integral may be evaluated with the help of the known formula[9]

$$\int_0^\infty dyy^s\,\mathrm{sech}^2 y=2^{1-s}(1-2^{1-s})\Gamma(1+s)\zeta(s), \quad \text{(D3)}$$

where $\zeta(s)$ is the Riemann zeta function. Differentiate (D3) with respect to s and then set $s=0$:

$$\int_0^\infty dy\ln y\,\mathrm{sech}^2 y=\ln\pi-2\ln 2-\gamma, \quad \text{(D4)}$$

where use has been made of the facts $\zeta(0)=-\frac{1}{2}$ and $\zeta'(0)=-\frac{1}{2}\ln(2\pi)$. The substitution of (D4) into (D2) gives (5.6).

APPENDIX E

In this Appendix, we generalize the considerations of this paper to the potential $-gr^{-\lambda}\ln r$, where $\lambda>3$. The generalization is straightforward in principle, but somewhat involved in the technical details. The same notation is used for arbitrary λ as for the special case $\lambda=4$.

(a) We repeat first the procedure of Appendix A. The radial differential equation is

$$d^2\psi/dr^2+gr^{-\lambda}(\ln r)\psi=0. \quad \text{(E1)}$$

With x and φ defined as before, $\varphi(x)$ satisfies

$$d^2\varphi/dx^2-gx^{\lambda-4}(\ln x)\varphi=0. \quad \text{(E2)}$$

The boundary conditions are not changed. Again let $x=\tau y$, but with τ chosen to satisfy

$$g\tau^{\lambda-2}\ln\tau=1, \quad \text{(E3)}$$

then

$$d^2\varphi/dy^2-y^{\lambda-4}(1+\epsilon\ln y)\varphi=0. \quad \text{(E4)}$$

Let

$$\nu=(\lambda-2)^{-1}, \quad \text{(E5)}$$

then (E4) can be converted into the integral equation

$$\varphi(y)=2\nu^\nu[\Gamma(\nu)]^{-1}y^{1/2}K_\nu(2\nu y^{\lambda/2-1})-2\nu\epsilon\int_0^\infty dy'(yy')^{1/2}I_\nu(2\nu y_<^{\lambda/2-1})\times K_\nu(2\nu y_>^{\lambda/2-1})\varphi(y')y'^{\lambda-4}\ln y', \quad \text{(E6)}$$

where I and K are the modified Bessel functions, and $y_<$ and $y_>$ are, respectively, the smaller and the larger one of y and y'. The first terms of the right-hand side of (A9) are generalized to

$$A=-(g/\epsilon)^\nu\nu^{2\nu}[\Gamma(1+\nu)]^{-1}\Gamma(1-\nu)\{1-\epsilon\nu^2[2\ln\nu+2-\psi(1+\nu)-\psi(1-\nu)]+O(\epsilon^2)\}, \quad \text{(E7)}$$

where ψ is the logarithmic derivative of the gamma function

$$\psi(z)=(d/dz)\ln\Gamma(z). \quad \text{(E8)}$$

In the derivation of (E7), the following integral is used

$$\int_0^\infty[K_\nu(t)]^2t^{-\rho}dt=2^{-2-\rho}[\Gamma(1-\rho)]^{-1}[\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\rho)]^2\times\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\rho+\nu)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\rho-\nu) \quad \text{(E9)}$$

for $1-\rho\pm2\nu>0$. In fact, (E9) is a special case of the Weber-Schafheitlin integral. For $\nu=\frac{1}{2}$, (E7) reduces to (A9) because of Gauss' theorem on $\psi(z)$.[10]

(b) Analogous to (2.7)–(2.10), we define

$$\psi_{n+1}(r,\Lambda)=g\int_{\Lambda^{-1}}^\infty dr'r'^\lambda\ln r'\min(r,r')\psi_n(r',\Lambda), \quad \text{(E10)}$$

$$A_n(\Lambda)=\psi_n(\infty,\Lambda)=g\int_{\Lambda^{-1}}^\infty dr'r'^{-\lambda+1}(\ln r')\psi_n(r',\Lambda), \quad \text{(E11)}$$

$$A_n(\Lambda)=g^n\Lambda^{n(\lambda-2)-1}\sum_{m=0}^n a_{nm}(\ln\Lambda)^{n-m}, \quad \text{(E12)}$$

and

$$B_m(\Lambda)=\sum_{n=m}^\infty a_{nm}g^n\Lambda^{n(\lambda-2)-1}(\ln\Lambda)^{n-m}. \quad \text{(E13)}$$

The Jost function $F(x)$ satisfies

$$F(x)-g\int_0^x dx'(x-x')x'^{\lambda-4}\ln x'F(x')=x. \quad \text{(E14)}$$

With $F_n(x)$ defined in the obvious manner, it is found

[9] Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, p. 32.

[10] See, for example, p. 19 of Ref. 9.

that for large x

$$F_n(x)=g^n x^{n(\lambda-2)+1}(\ln x)^n \nu^{2n}[\Gamma(n+1+\nu)n!]^{-1} \times[1+O((\ln x)^{-1})]. \quad (E15)$$

Thus, in the same sence as (3.16),

$$F(x)\sim\Gamma(1+\nu)x(\tfrac{1}{2}z)^{-\nu}I_\nu(z), \quad (E16)$$

where

$$z=2(g\nu^2x^{\lambda-2}\ln x)^{1/2}. \quad (E17)$$

Furthermore, (3.18) is replaced by

$$F'(x)=dF(x)/dx=\partial F(x)/\partial x + (2\nu)^{-1}z[1+\nu(\ln x)^{-1}]x^{-1}\partial F(x)/\partial z, \quad (E18)$$

and

$$F'(x)\sim\Gamma(\nu)(\tfrac{1}{2}z)^{-\nu+1}I_{\nu-1}(z). \quad (E19)$$

Since

$$\int dz z^{-1}[I_{\nu-1}(z)]^{-2}=-K_{\nu-1}(z)/I_{\nu-1}(z), \quad (E20)$$

we get immediately that

$$B_0(\Lambda)=2\nu^{2\nu-1}[\Gamma(\nu)]^{-2}(g\ln\Lambda)^\nu K_{\nu-1}(\Omega)/I_{\nu-1}(\Omega), \quad (E21)$$

where

$$\Omega=2(g\nu^2\Lambda^{\lambda-2}\ln\Lambda)^{1/2}, \quad (E22)$$

and that as $\Lambda\to\infty$,

$$B_n(\Lambda)=(-1)^{n+1}\nu^{2\nu}[\Gamma(1+\nu)]^{-1} \times\Gamma(1-\nu)C_n^{\nu}(g\ln\Lambda)^\nu+o(1), \quad (E23)$$

where C_n^ν is as before the binomial coefficient.

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Charged-Pion Photoproduction from Deuterium with Polarized Bremsstrahlung*

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Measurements have been made on the ratio of pion-production cross sections at right angles to and along the photon electric-field vector. The positive and negative pions were first momentum-analyzed and counted by means of a counter telescope. Data have been taken at 45, 90, and 135° in the c.m. system, and at proton energies of 225, 330, and 450 MeV. A comparison of the data is made with the dispersion-relation calculation of McKinley.

I. INTRODUCTION

THE photoproduction of positive and negative pions from deuterium has been extensively studied in the energy region from threshold to 500 MeV.[1] In all previous experiments, either the total cross sections or the angular distributions were observed. The present experiment concerns the asymmetry of the pions photoproduced by polarized gamma rays. The positive pion production from polarized gamma rays has been studied in this laboratory[2] and the present experiment is an extension to the study of negative pion production from deuterium. Measurements were made at photon energies of 225, 330, and 450 MeV.

The production asymmetry A is defined as $(\sigma_\perp-\sigma_{\parallel})/(\sigma_\perp+\sigma_{\parallel})$, where $\sigma_\perp$ and $\sigma_{\parallel}$ refer to the meson-production cross section perpendicular and parallel to the plane of polarization of the photon. The measurement of A for positive pion production from hydrogen has shown some disagreement with the dispersion-relation calculations, and no reasonable set of pion-nucleon phase shifts can make those calculations compatible with the observed angular behaviors of the asymmetry. Moreover, the introduction of $\gamma\pi\rho$ coupling does not improve the agreement appreciably. The present experiment shows the same discrepancy between the theory and the measured values. The measurements were made at energies sufficiently remote from the pion-production threshold that the final-state interaction can reasonably be neglected. For photon energies between 200 and 500 MeV, the analysis of meson production from deuterium in terms of free-nucleon cross sections has been demonstrated to be satisfactory.

II. EXPERIMENTAL METHOD

A polarized bremsstrahlung beam, developed by Taylor and Mozley,[2] was produced by placing a thin (0.003-in.) aluminum foil at the end of the Stanford Mark III linear accelerator. A beam of electrons striking the foil produced bremsstrahlung polarized perpendicular to the plane of emission. The polarization is a function of the angle which the photon makes with the initial direction of the electron, reaching a maximum at an angle of mc^2/E_0, where E_0 is the initial electron energy. We have calculated the polarization

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[1] D. H. White, R. M. Schectman, and B. M. Chasan, Phys. Rev. 120, 614 (1960), and references therein.

[2] R. E. Taylor and R. F. Mozley, Phys. Rev. 117, 835 (1960); R. C. Smith and R. F. Mozley, *ibid.* 130, 2429 (1963).